$$= t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \ldots\ldots \qquad \left[\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\right]$$

$$L(\sin\sqrt{t}) = L\left(t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \ldots\ldots\right)$$

$$= \frac{\Gamma 3/2}{s^{3/2}} - \frac{\Gamma 5/2}{3!s^{5/2}} + \frac{\Gamma 7/2}{5!s^{7/2}} - \ldots\ldots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left\{1 - \left(\frac{1}{2^2 s}\right) + \frac{1}{2!}\left(\frac{1}{2^2 s}\right)^2 - \ldots\ldots\right\}$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left\{1 - \left(\frac{1}{2^2 s}\right) + \frac{1}{2!}\left(\frac{1}{2^2 s}\right)^2 - \ldots\ldots\right\}$$

$$\left[e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\right]$$

$$\Rightarrow \quad L(\sin\sqrt{t}) = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-(1/4s)}$$

Now, $L\left[\frac{d}{dt}(\sin\sqrt{t})\right] = s\,L(\sin\sqrt{t}) - 0 \qquad \left[\because F(0) = 0 \text{ and } L\left[\frac{d}{dt}[F(t)]\right] = sF(s)\right]$

$$L\left(\frac{\cos\sqrt{t}}{2\sqrt{t}}\right) = \frac{\sqrt{\pi}}{2\sqrt{s}} e^{-\left(\frac{1}{4s}\right)}$$

Ans.

10.9 LAPLACE TRANSFORM OF INTEGRAL OF f(t)

$$\boxed{L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}F(s)} \qquad \text{where} \quad L[f(t)] = F(s)$$

Proof. Let $\phi(t) = \int_0^t f(t)\,dt$ and $\phi(0) = 0$ then $\phi'(t) = f(t)$

We know the formula of Laplace transforms of $\phi'(t)$ *i.e.*

$$L[\phi'(t)] = s\,L[\phi(t)] - \phi(0)$$

$$\Rightarrow \quad L[\phi'(t)] = s\,L[\phi(t)] \qquad [\phi(0) = 0]$$

$$\Rightarrow \quad L[\phi(t)] = \frac{1}{s}L[\phi'(t)]$$

Putting the values of $\phi(t)$ and $\phi'(t)$, we get

$$L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}L[f(t)] \Rightarrow \boxed{L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}F(s)}$$

Proved.

Note. (1) Laplace transform of **Integral** of $f(t)$ corresponds to the division of the Laplace transform of $f(t)$ by s.

(2) $$\int_0^t f(t)\,dt = L^{-1}\left[\frac{1}{s}F(s)\right]$$

10.10 LAPLACE TRANSFORM OF $t.f(t)$ (Multiplication by t)

If $L[f(t)] = F(s)$, then

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)].$$ (U.P., II Semester, Summer 20..)

Proof. $L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\, dt$

Differentiating (1) w.r.t. 's', we get

$$\frac{d}{ds}[F(s)] = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)\, dt\right] = \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st}\right) f(t)\, dt$$

$$= \int_0^\infty \left(-t e^{-st}\right).f(t)\, dt = \int_0^\infty e^{-st}\left[-t.f(t)\right] dt$$

$$= L\left[-t f(t)\right] \Rightarrow L\left[t f(t)\right] = (-1)^1 \frac{d}{ds}\left[F(s)\right]$$

Similarly, $L\left[t^2 f(t)\right] = (-1)^2 \frac{d^2}{ds^2}\left[F(s)\right]$

$$L\left[t^3 f(t)\right] = (-1)^3 \frac{d^3}{ds^3}\left[F(s)\right]$$

$$\boxed{L\left[t^n f(t)\right] = (-1)^n \frac{d^n}{ds^n}\left[F(s)\right]}$$

10.11 INITIAL AND FINAL VALUE THEOREMS

(a) Initial Value Theorem. $L\{f(t)\} = F(s)$

$\Rightarrow$ $\lim_{t \to 0} f(t) = \lim_{s \to \infty}\left[sF(s)\right]$, provided the limit exists.

Proof. $L\{f'(t)\} = s\,L\{f(t)\} - f(0)$

$\Rightarrow$ $\int_0^\infty e^{-st} f'(t)\, dt = s\,F(s) - f(0)$

$\Rightarrow$ $\lim_{s \to \infty} \int_0^\infty e^{-st} f'(t)\, dt = \lim_{s \to \infty}\left[sF(s) - f(0)\right]$

$\Rightarrow$ $\lim_{s \to \infty}\left[s\,F(s)\right] = f(0) + \int_0^\infty \left(\lim_{s \to \infty} e^{-st}\right) f'(t)\, dt$

$$= f(0) + \int_0^\infty (0)\, f'(t)\, dt \qquad (\because \lim_{s \to \infty} e^{-st} ...$$

$$= f(0) + 0 = f(0) = \lim_{t \to 0} f(t)$$

(b) Final Value Theorem. $L\{f(t)\} = F(s)$

$\Rightarrow$ $\lim_{t \to \infty} f(t) = \lim_{s \to 0}\left[sF(s)\right]$, provided the limits exist.

Proof. $L\{f'(t)\} = sL\{f(t)\} - f(0) \Rightarrow \int_0^\infty e^{-st} f'(t)\, dt = sF(s) - f(0)$

$$\lim_{s\to 0}\int_0^{\infty} e^{-st} f'(t)\,dt = \lim_{s\to 0}\left[sF(s)-f(0)\right]$$

$\Rightarrow$
$$\lim_{s\to 0}[s\,F(s)-f(0)] = \lim_{s\to 0}\int_0^{\infty} e^{-st} f'(t)\,dt$$

$\Rightarrow$
$$\lim_{s\to 0}[s\,F(s)]-f(0) = \lim_{s\to 0}\int_0^{\infty} e^{-st} f'(t)\,dt$$

$\Rightarrow$
$$\lim_{s\to 0}\left[sF(s)\right]-f(0) = \int_0^{\infty}\lim_{s\to 0} e^{-st} f'(t)\,dt = \int_0^{\infty}(1)\,f'(t)\,dt \qquad \left[\because \lim_{s\to 0} e^{-st}=1\right]$$

$\Rightarrow$
$$\boxed{\lim_{s\to 0}\left[sF(s)\right] = \lim_{t\to\infty} f(t)}$$

Example 17. *If* $L\{F(t)\} = \dfrac{1}{s(s+\beta)}$ *then, find* $\lim\limits_{t\to\infty} F(t)$

Solution. By final-value theorem,

$$\lim_{t\to\infty} F(t) = \lim_{s\to 0} sL\{F(t)\} = \lim_{s\to 0}\frac{s}{s\,(s+\beta)} = \lim_{s\to 0}\frac{1}{(s+\beta)} = \frac{1}{\beta} \qquad \textbf{Ans.}$$

10.12 EXPONENTIAL INTEGRAL FUNCTION $\int_t^{\infty}\left(\frac{e^{-x}}{x}\right)dx$

Let $\quad f(t) = \int_t^{\infty}\frac{e^{-x}}{x}\,dx$

$\Rightarrow \quad f'(t) = -\dfrac{e^{-t}}{t} \quad \Rightarrow \quad tf'(t) = -e^{-t}$ [Here – ve sign appears due to lower limit]

Taking Laplace Transform of $tf'(t)$, we get

$$L\{t\,f'(t)\} = L\left\{-e^{-t}\right\} = -L\left\{e^{-t}\right\}$$

$\Rightarrow$
$$-\frac{d}{ds}\left[sF(s)-f(0)\right] = -\frac{1}{s+1}$$

$\Rightarrow$
$$\frac{d}{ds}\left[s\,F(s)\right] = \frac{1}{s+1} \qquad \left[\because f(0)=\text{constant} \therefore \frac{d}{ds}f(0)=0\right]$$

Integrating both the sides, we get

$$s\,F(s) = \log(s+1)+C \qquad ...(1)$$

Now, by final value theorem, we have

$$\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t) \qquad ...(2)$$

Hence, $\lim\limits_{s\to 0}\left[sF(s)\right] = \lim\limits_{s\to 0}[\log(s+1)+C] = 0+C=C \qquad ...(3)$

Also,
$$\lim_{t\to\infty}\left[f(t)\right] = \lim_{t\to\infty}\int_t^{\infty}\left(\frac{e^{-x}}{x}\right)dx = 0 \qquad ...(4)$$

Putting the values of $\lim_{s \to 0} [s\, F(s)]$ and $\lim_{t \to \infty}[f(t)]$ from (3) and (4) in (2), we get

$C = 0.$

Hence from (1), $sF(s) = \log(s+1) \Rightarrow F(s) = \left\{\frac{\log(s+1)}{s}\right\}$

$$\Rightarrow \quad \boxed{L\int_t^{\infty}\left(\frac{e^{-x}}{x}\right)dx = \left[\frac{\log(s+1)}{s}\right]}$$

Example 18. *Find the Laplace Transform of t sin at.*

Solution.

$$L(t \sin at) = L\left(t\frac{e^{iat}-e^{-iat}}{2i}\right) = \frac{1}{2i}[L(t.e^{iat}) - L(te^{-iat})]$$

$$= \frac{1}{2i}\left[-\frac{d}{ds}\frac{1}{s-ia} + \frac{d}{ds}\frac{1}{s+ia}\right]$$

$$\frac{1}{2i}\left[\frac{1}{(s-ia)^2} - \frac{1}{(s+ia)^2}\right] = \frac{1}{2i}\left[\frac{(s+ia)^2-(s-ia)^2}{(s-ia)^2(s+ia)^2}\right]$$

$$= \frac{1}{2i}\frac{(s^2+2ias-a^2)-(s^2-2ias-a^2)}{(s^2+a^2)^2}.$$

$$= \frac{1}{2i}\frac{4ias}{(s^2+a^2)^2} = \frac{2as}{(s^2+a^2)^2}$$

Example 19. *Find the Laplace transform of t sinh at.*

Solution.

$$L(\sinh at) = \frac{a}{s^2-a^2}$$

$$L[t \sinh at] = -\frac{d}{ds}\left(\frac{a}{s^2-a^2}\right)$$

$$\Rightarrow \quad = \frac{2as}{(s^2-a^2)^2}$$

Example 20. *Find the Laplace transform of t^2 cos at*

Solution.

$$L(\cos at) = \frac{s}{s^2+a^2}$$

$$L(t^2 \cos at) = (-1)^2\frac{d^2}{ds^2}\left[\frac{s}{s^2+a^2}\right] = \frac{d}{ds}\frac{(s^2+a^2).1-s(2s)}{(s^2+a^2)^2} = \frac{d}{ds}\frac{a^2-s^2}{(s^2+a^2)^2}$$

$$= \frac{(s^2+a^2)^2(-2s)-(a^2-s^2).2(s^2+a^2)(2s)}{(s^2+a^2)^4} = \frac{(s^2+a^2)(-2s)-(a^2-s^2)...}{(s^2+a^2)^3}$$

$$= \frac{-2s^3-2a^2s-4a^2s+4s^3}{(s^2+a^2)^3} = \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}$$

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Example 21. *Obtain the Laplace transform of* $t^2 e^t . \sin 4t$. *(U.P. II Semester, Summer 2002)*

Solution. $L(\sin 4t) = \dfrac{4}{s^2+16}$,

$$L\left(e^t \sin 4t\right) = \frac{4}{(s-1)^2+16}$$

$$L\left(t e^t \sin 4t\right) = -\frac{d}{ds}\left(\frac{4}{s^2-2s+17}\right) = \frac{4(2s-2)}{\left(s^2-2s+17\right)^2}$$

$$L\left(t^2 e^t \sin 4t\right) = -\frac{d}{ds}\left(\frac{4(2s-2)}{\left(s^2-2s+17\right)^2}\right)$$

$$= -4\frac{\left(s^2-2s+17\right)^2 2-(2s-2)2\left(s^2-2s+17\right)(2s-2)}{\left(s^2-2s+17\right)^4}$$

$$= -4\frac{(s^2-2s+17)\,2-2(2s-2)^2}{(s^2-2s+17)^3}$$

$$= \frac{-4\left(2s^2-4s+34-8s^2+16s-8\right)}{\left(s^2-2s+17\right)^3}$$

$$= \frac{-4\left(-6s^2+12s+26\right)}{\left(s^2-2s+17\right)^3} = \frac{8\left[3s^2-6s-13\right]}{(s^2-2s+17)^3}$$ **Ans.**

Example 22. *Find the Laplace transform of the function*

$f(t) = te^{-t}\sin 2t$ *(U.P. II Semester, Summer 2002)*

Solution. $L[\sin 2t] = \dfrac{2}{s^2+4}$

$$L\left[e^{-t}\sin 2t\right] = \frac{2}{(s+1)^2+4} = F(s) \quad \text{(say)}$$

$$L\left(te^{-t}\sin 2t\right) = -F'(s) = -\frac{d}{ds}\left[\frac{2}{(s+1)^2+4}\right] = \frac{2\cdot 2(s+1)}{\left[(s+1)^2+4\right]^2}$$

$$= \frac{4(s+1)}{\left[(s+1)^2+4\right]^2}$$ **Ans.**

EXERCISE 10.2

Find the Laplace transforms of the following :

1. $t \cosh at$ **Ans.** $\dfrac{s^2+a^2}{\left(s^2-a^2\right)^2}$

2. $t \cos t$ Ans. $\dfrac{s^2-1}{\left(s^2+1\right)^2}$

3. $t \cosh t$ Ans. $\dfrac{s^2+1}{\left(s^2-1\right)^2}$

4. $t^2 \sin t$ Ans. $\dfrac{2\left(3s^2-1\right)}{\left(s^2+1\right)^3}$

5. $t^3 e^{-3t}$ Ans. $\dfrac{6}{(s+3)^4}$

6. $t \sin^2 3t$ Ans. $\dfrac{1}{2}\left[\dfrac{1}{s^2}-\dfrac{s^2-36}{\left(s^2+36\right)^2}\right]$

7. $t e^{at} \sin at$ Ans. $\dfrac{2a(s-a)}{\left(s^2-2as+2a^2\right)^2}$

8. $te^{-t} \cosh t$ Ans. $\dfrac{s^2+2s+2}{\left(s^2+2s\right)^2}$

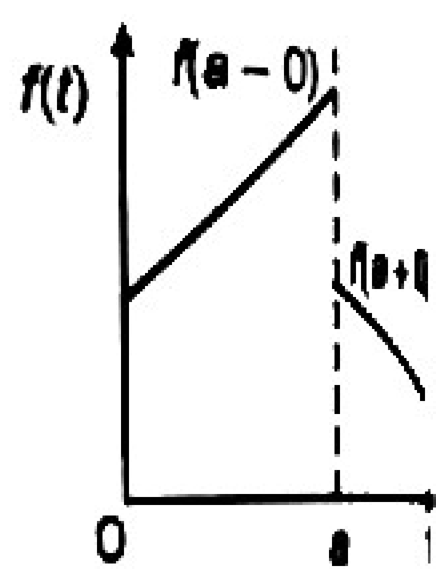
9. $t^2 e^{-2t} \cos t$ Ans. $\dfrac{2(s^3+6s^2+9s+2)}{(s^2+4s+5)^3}$

10. $\int_0^t e^{-2t} t \sin^3 t \, dt$ Ans. $\dfrac{3(s+2)}{2s}\left[\dfrac{1}{\left[(s+2)^2+9\right]^2}-\dfrac{1}{\left[(s+2)^2+1\right]^2}\right]$

11. If $f(t)$ is continuous, except for an ordinary discontinuity at $t = a$, $(a > 0)$ as given in the figure, then show that

$$L\left[f'(t)\right]=s\left[f(t)\right]-f(0)-e^{-as}\left[f(a+0)-f(a-0)\right]$$

(U.P. II Semester 2003)



12. Pick the correct statement for final value theorem of Laplace transform:

(i) $\underset{t\to 0}{\text{Lt}}\ f(t)=\underset{s\to\infty}{\text{Lt}}\ s\,F(s)$ (ii) $\underset{t\to\infty}{\text{Lt}}\ f(t)=\underset{s\to 0}{\text{Lt}}\ s\,F(s)$

(U.P. II Semester 2010) Ans.

10.13 LAPLACE TRANSFORM OF $\frac{1}{t}f(t)$ (Division by t)

If $L[f(t)]=F(s)$, then $L\left[\dfrac{1}{t}f(t)\right]=\int_s^\infty F(s)\,ds$ *(U.P. II Semester Summer, 2007, 200…)*

Proof. We know that $L\left[f(t)\right]=F(s)$ or $F(s)=\int_0^\infty e^{-st} f(t)\,dt$

Integrating (1) w.r.t. 's', we have

$$\int_s^\infty F(s)\,ds = \int_s^\infty\left[\int_0^\infty e^{-st} f(t)\,dt\right]ds$$

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\,ds\right]dt = \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt$$

$$= \int_0^\infty \frac{-f(t)}{t}\left[e^{-st}\right]_s^\infty dt = \int_0^\infty \frac{-f(t)}{t}\left[0-e^{-st}\right]dt$$

$$= \int_0^\infty e^{-st}\left\{\frac{1}{t}\cdot f(t)\right\}dt = L\left[\frac{1}{t}f(t)\right]$$

$$\Rightarrow \boxed{L\left[\frac{1}{t}f(t)\right]=\int_s^\infty F(s)\,ds}$$

Cor. $$L^{-1}\int_s^{\infty} F(s)\, ds = \frac{1}{t} f(t)$$

Example 23. *Find the Laplace transform of* $\frac{\sin 2t}{t}$

Solution. $L(\sin 2t) = \frac{2}{s^2+4}$

$$L\left(\frac{\sin 2t}{t}\right) = \int_s^{\infty} \frac{2}{s^2+4}\, ds = 2.\frac{1}{2}\left[\tan^{-1}\frac{s}{2}\right]_s^{\infty}$$

$$= \left[\tan^{-1}\infty - \tan^{-1}\frac{s}{2}\right] = \frac{\pi}{2} - \tan^{-1}\frac{s}{2}$$

$$= \cot^{-1}\frac{s}{2}$$

Ans.

Example 24. *Find the Laplace transform of* $f(t) = \int_0^t \frac{\sin at}{t}\, dt$

(M.D.U., Dec. 2009, U.P., II Semester, Summer 2005)

Solution. $$L(\sin at) = \frac{a}{s^2+a^2}$$

$$L\left(\frac{\sin at}{t}\right) = \int_s^{\infty} \frac{a}{s^2+a^2}\, ds = \left[\tan^{-1}\frac{s}{a}\right]_s^{\infty} = \frac{\pi}{2} - \tan^{-1}\frac{s}{a} = \cot^{-1}\frac{s}{a}$$

Hence, $$L\left[\int_0^t \frac{\sin at}{t}\, dt\right] = \frac{1}{s}\cot^{-1}\frac{s}{a}$$

Ans.

Example 25. *Find the Laplace transform of:*

$$\frac{\cos at - \cos bt}{t}$$

(R.G.P.V., Bhopal, Dec. 2010)
(Uttarakhand, II Semester, June 2007)
(U.P., II-Semester, 2004)

Solution. Here, $$f(t) = \frac{\cos at - \cos bt}{t}$$

We know that, $$L(\cos at - \cos bt) = L(\cos at) - L(\cos bt) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$L\left(\frac{\cos at - \cos bt}{t}\right) = \int_s^{\infty}\left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^{\infty} = \frac{1}{2}\left[\log\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right]_s^{\infty}$$

$$= \frac{1}{2}\log 1 - \frac{1}{2}\log\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}} = 0 - \frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2} \quad [\log 1 = 0]$$

$$= \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

Ans.

Example 26. *If* $f(t)=\dfrac{e^{at}-\cos bt}{t}$, *find the Laplace transform of* $f(t)$.

(*U.P. II Semester, Summer* ...)

Solution. $f(t)=\dfrac{e^{at}-\cos bt}{t}=\dfrac{e^{at}}{t}-\dfrac{\cos bt}{t}$

We know that, $L\left(e^{at}-\cos bt\right)=\left(\dfrac{1}{s-a}-\dfrac{s}{s^2+b^2}\right)$

$$\therefore \quad L\left(\frac{e^{at}-\cos bt}{t}\right)=\int_s^{\infty}\left(\frac{1}{s-a}-\frac{s}{s^2+b^2}\right)ds=\left[\log(s-a)-\frac{1}{2}\log\left(s^2+b^2\right)\right]_s^{\infty}$$

$$=\left[\frac{2\log(s-a)-\log\left(s^2+b^2\right)}{2}\right]_s^{\infty}=\frac{1}{2}\left[\log(s-a)^2-\log\left(s^2+b^2\right)\right]_s^{\infty}$$

$$=\frac{1}{2}\left[\log\frac{(s-a)^2}{s^2+b^2}\right]_s^{\infty}=\frac{1}{2}\left[\log\left\{\frac{\left(1-\frac{a}{s}\right)^2}{1+\frac{b^2}{s^2}}\right\}\right]_s^{\infty}$$

$$=\frac{1}{2}\left[0-\log\frac{\left(1-\frac{a}{s}\right)^2}{\left(1+\frac{b^2}{s^2}\right)}\right]=\frac{1}{2}\left[\log\frac{s^2+b^2}{(s-a)^2}\right]$$

Example 27. *Find the Laplace transform of* $\dfrac{1-\cos t}{t^2}$.

Solution. $L(1-\cos t)=L(1)-L(\cos t)=\dfrac{1}{s}-\dfrac{s}{s^2+1}$

$$L\left[\frac{1-\cos t}{t}\right]=\int_s^{\infty}\left(\frac{1}{s}-\frac{s}{s^2+1}\right)ds=\left[\log s-\frac{1}{2}\log\left(s^2+1\right)\right]_s^{\infty}$$

$$=\frac{1}{2}\left[\log s^2-\log\left(s^2+1\right)\right]_s^{\infty}=\frac{1}{2}\left[\log\frac{s^2}{s^2+1}\right]_s^{\infty}$$

$$=\frac{1}{2}\left[\log\frac{1}{\left(1+\frac{1}{s^2}\right)}\right]_s^{\infty}=\frac{1}{2}\left[0-\log\frac{s^2}{s^2+1}\right]=-\frac{1}{2}\log\frac{s^2}{s^2+1}$$

Again. $L\left[\frac{1-\cos t}{t^2}\right]=-\frac{1}{2}\int_s^\infty \log\frac{s^2}{s^2+1}\,ds=-\frac{1}{2}\int_s^\infty\left(\log\frac{s^2}{s^2+1}.1\right)ds$

Integrating by parts, we have,

$$=-\frac{1}{2}\left[\log\frac{s^2}{s^2+1}.s-\int\frac{s^2+1}{s^2}\frac{(s^2+1)2s-s^2(2s)}{(s^2+1)^2}.sds\right]_s^\infty$$

$$=-\frac{1}{2}\left[s\log\frac{s^2}{s^2+1}-2\int\frac{1}{s^2+1}ds\right]_s^\infty=-\frac{1}{2}\left[s\log\frac{s^2}{s^2+1}-2\tan^{-1}s\right]_s^\infty$$

$$=-\frac{1}{2}\left[0-2\left(\frac{\pi}{2}\right)-s\log\frac{s^2}{s^2+1}+2\tan^{-1}s\right]=-\frac{1}{2}\left[-\pi-s\log\frac{s^2}{s^2+1}+2\tan^{-1}s\right]$$

$$=\frac{\pi}{2}+\frac{s}{2}\log\frac{s^2}{s^2+1}-\tan^{-1}s$$

$$=\left(\frac{\pi}{2}-\tan^{-1}s\right)+\frac{s}{2}\log\frac{s^2}{s^2+1}=\cot^{-1}s+\frac{s}{2}\log\frac{s^2}{s^2+1}$$ **Ans.**

Example 28. *Evaluate* $L\left[e^{-4t}\frac{\sin 3t}{t}\right]$

Solution. $L[\sin 3t]=\frac{3}{s^2+3^2}$

$$\Rightarrow\quad L\left[\frac{\sin 3t}{t}\right]=\int_s^\infty\frac{3}{s^2+9}ds=\left[\frac{3}{3}\tan^{-1}\frac{s}{3}\right]_s^\infty=\left[\tan^{-1}\frac{s}{3}\right]_s^\infty$$

$$=\frac{\pi}{2}-\tan^{-1}\frac{s}{3}=\cot^{-1}\frac{s}{3}$$

$$L\left[e^{-4t}\frac{\sin 3t}{t}\right]=\cot^{-1}\frac{s+4}{3}=\tan^{-1}\frac{3}{s+4}$$ **Ans.**

EXERCISE 10.3

Find Laplace transform of the following:

1. $\frac{1}{t}(1-e^t)$ **Ans.** $\log\frac{s-1}{s}$
2. $\frac{1}{t}\left(e^{-at}-e^{-bt}\right)$ **Ans.** $\log\frac{s+b}{s+a}$
3. $\frac{1}{t}(1-\cos at)$ **Ans.** $-\frac{1}{2}\log\frac{s^2}{s^2+a^2}$
4. $\frac{1}{t}\sin^2 t$ **Ans.** $\frac{1}{4}\log\frac{s^2+4}{s^2}$
5. $\frac{1}{t}\sinh t$ **Ans.** $-\frac{1}{2}\log\frac{s-1}{s+1}$
6. $\frac{1}{t}\left(e^{-t}\sin t\right)$ **Ans.** $\cot^{-1}(s+1)$
7. $\frac{1}{t}(1-\cos t)$ **Ans.** $\frac{1}{2}\left[\log\left(s^2+1\right)-\log s\right]$
8. $\int_0^\infty\frac{1}{t}e^{-2t}\sin t\,dt$ **Ans.** $\frac{1}{s}\cot^{-1}(s+2)$
9. $\int_0^\infty\frac{e^{-t}-e^{-3t}}{t}dt$ **Ans.** $\log 3$